Therefore, $\phi(\theta_1) \leq \phi(\theta_2)$.

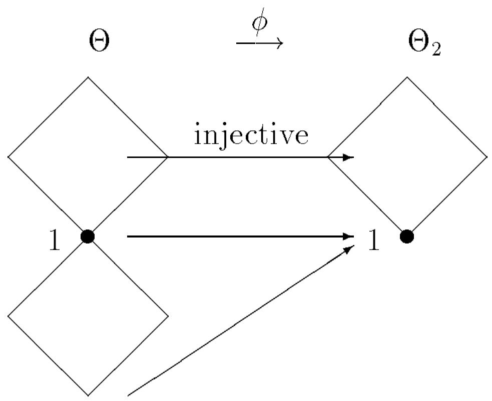
Given any f, take $B, C = 1$ in the definition of O_2 to verify $1 \in O_2(f)$, that is $\Theta_2(1) \leq \theta$ for any $\theta \in \Theta_2$. Because the map is order-preserving, if $\theta \leq \Theta(1)$ then $\phi(\theta) \leq \Theta_2(1)$. However $\Theta_2(1) \leq \phi(\theta)$. So $\phi(\theta) = \Theta_2(1)$.

Select the function $1(n) = 1$ from the class $1 \in \Theta$. The statement $\theta_1, \theta_2 \geq 1$ translates to:

$$\forall f \in \theta_i \; \exists C, B > 0 \text{ s.t. } \forall n > B, \; 1 < Cf(n).$$

We can therefore select from θ_i an f_i for which $f(n) > 1$ for all n. In which case $\max(f_i(n), 1) = f_i(n)$ for all n. Hence if $f_1 \leq f_2$ then $f_1 \leq f_2$ and if $f_2 \leq f_1$ then $f_2 \leq f_1$. Putting this together, if $\Theta_2(f_1) = \Theta_2(f_2)$ then $\Theta(f_1) = \Theta(f_2)$. This proves the remark about injectivity. $\square$

In summary, the algebraic approach sums up the difference between O and O_2 definitions by the following picture:



The relation $\leq$ on $\Theta(\mathcal{F})$ is reflexive: for any $\theta \in \Theta(\mathcal{F})$, take any $f \in \theta$. Since $f \leq f$ then $\theta \leq \theta$.

The relation $\leq$ on $\Theta(\mathcal{F})$ is transitive: let $\theta_1, \theta_2, \theta_3 \in \Theta(\mathcal{F})$ be such that $\theta_1 \leq \theta_2$ and $\theta_2 \leq \theta_3$. There exist $f_1 \in \theta_1, f_2 \in \theta_2$, and $f_3 \in \theta_3$ such that $f_1 \leq f_2$ and $f_2 \leq f_3$. Therefore $f_1 \leq f_3$ hence $\theta_1 \leq \theta_3$.

The relation $\leq$ on $\Theta(\mathcal{F})$ is anti-symmetric: let θ_1 and θ_2 be elements of $\Theta(\mathcal{F})$ for which $\theta_1 \leq \theta_2$ and $\theta_2 \leq \theta_1$. Since for all f_1 taken from θ_1 and all f_2 taken from θ_2, both $f_1 \leq f_2$ and $f_2 \leq f_1$, all such pairs are theta-equivalent. This means that any f_1 in θ_1 is also in θ_2, and vice-a-versa. So $\theta_1 = \theta_2$. $\square$

In order to make the notation less cumbersome, we shall simply write Θ for $\Theta(\mathcal{F})$. We have a special interest in the class $\Theta(1)$ in Θ and give it the name 1. Starting from preorder O_2 we can construct a partial order Θ_2 in the exact same manner that Θ was constructed from O. The function $\Theta_2(f)$ will take an $f \in \mathcal{F}$ to its equivalence class in Θ_2. Its class $\Theta_2(1)$ will also be denoted 1. Which 1 will be discernible from context.

Let A and B be two partial orders whose ordering is denoted generically by $\leq$. A map $f : A \to B$ is *order-preserving* if,

$$a \leq b \;\Rightarrow\; f(a) \leq f(b).$$

Fact 5 *The following is a well-defined order-preserving map:*

$$\begin{array}{rccl} \phi: & \Theta & \to & \Theta_2 \\ & \theta & \mapsto & \Theta_2(f), \ \textit{for any } f \in \theta. \end{array}$$

And claim:

1. *For any $\theta \in \Theta_2$, $1 \leq \theta$. That is, 1 is the smallest element in Θ_2.*
2. *For all $\theta \in \Theta$ such that $\theta \leq 1$, $\phi(\theta) = 1$.*
3. *The restriction of ϕ to $\{\, \theta \in \Theta \mid \theta \geq 1 \,\}$ is injective.*

PROOF: Well-definedness follows from $O(f) \subseteq O_2(f)$: let f and g be from θ. Then $f \in O(g) \;\Rightarrow\; f \in O_2(g)$ and $g \in O(f) \;\Rightarrow\; g \in O_2(f)$, so f and g are theta-equivalent by Θ_2.

Take $\theta_1, \theta_2 \in \Theta$ such that $\theta_1 \leq \theta_2$. Choosing $f_1 \in \theta_1$ and $f_2 \in \theta_2$,

$$f_1 \in O(f_2) \;\Rightarrow\; f_1 \in O_2(f_2).$$

The algebraic viewpoint

There is another way to look at these results. Recall that a *preorder* is a relation $\leq$ such that,

1. The relation is reflexive: $a \leq a$,

2. The relation is transitive: $a \leq b$ and $b \leq c$ implies $a \leq c$.

The following is a preorder on $\mathcal{F}$:

$$f \leq g \iff f \in O(g).$$

For a preorder to be a *partial order*, anti-symmetry must hold,

$$a \leq b \text{ and } b \leq a \text{ implies } a = b,$$

in the sense that "they are the same object". Obviously, $\leq$ is not a partial order. Desiring a partial order, we make it one by a general technique:

Definition 2 *Two functions $f, g \in \mathcal{F}$ are said to be theta-equivalent if*

$$f \leq g \text{ and } g \leq f.$$

We write $\Theta(f)$ for the class of functions theta-equivalent to f, and $\Theta(\mathcal{F})$ for the set of all theta-equivalence classes, the image of $\mathcal{F}$ under Θ.

Fact 4 *Define the relation $\leq$ on $\Theta(\mathcal{F})$ as follows. Given any two elements θ_1 and θ_2 from $\Theta(\mathcal{F})$ and any two functions $f_1 \in \theta_1$ and $f_2 \in \theta_2$,*

$$\theta_1 \leq \theta_2 \iff f_1 \leq f_2.$$

Then $\leq$ is a partial order on $\Theta(\mathcal{F})$.

PROOF: First, we demonstrate the relation is well-defined. Suppose f_1 and f_2 are taken from θ_1 and θ_2, respectively, and that $f_1 \leq f_2$. If g_1 and g_2 are other choices from θ_1 and θ_2, respectively, then,

$$g_1 \leq f_1 \leq f_2 \leq g_2,$$

so $f_1 \leq f_2$ irrespective of the choice of test elements.

and for $n \leq B$,

$$C'g(n) \geq Dg(n) \geq (f(n)/g(n)+1)g(n) > f(n).$$

Hence $f \in O_1(g)$. □

Fact 2 *Let $g \in \mathcal{F}$. Define,*

$$O_2(g) = \{ f \in \mathcal{F} \mid \exists B, C > 0,\ \forall n > B,\ f(n) < C \max(g(n), 1) \}.$$

Then for every $g \in \mathcal{F}$ which is positive and nondecreasing on a certain interval $[a, \infty)$,

$$f \in O(g) \iff f \in O_2(g).$$

Proof: From $g(n) \leq \max(g(n), 1)$ follows $O(g) \subseteq O_2(g)$. Suppose $f \in O_2(g)$, and fix B and C as required by the definition of $O_2(g)$. Let,

$$C' = C \max(1, 1/g(a)).$$

For all $n \geq a$,

$$C'g(n) \geq Cg(n)/g(a) \geq C.$$

Therefore, for all $n > \max(a, B)$,

$$C'g(n) = \max(C'g(n), C) \geq \max(Cg(n), C) = C \max(g(n), 1) > f(n).$$

Hence, $f \in O(n)$. □

The positive and nondecreasing supposition is needed, as $1/n = O_2(1/n^2)$ however $1/n \neq O(1/n^2)$.

Fact 3 *For $f, g \in \mathcal{F}$, $f \in O(g) \Rightarrow f \in O_2(g)$.*

Proof: This is shown in the first sentence of the preceding proof. □

Remark: Typically, $f \in O(g)$ is written $f = O(g)$.

Notes on Big-Oh

DATE: MAY 22, 1993
BURT ROSENBERG

Introduction

The Big-Oh notation is utilized to analyze the run time of algorithms. The idea is to avoid considering algorithms which differ by "constant factors". The distinction should be maintained between algorithm and program. A program's run time depends on constant factors, an algorithm's does not.

Numerous variants of the Big-Oh notation can be proposed. Jan Plaza and I investigated the consequences of changing definitions. The second section attempts to understand the situation using the techniques of universal-algebra.

We take $\mathbf{N}$ to be the naturals, $0, 1, 2, \ldots$ and $\mathbf{R}$ to be the reals. We consider the space of functions $\mathcal{F} : \mathbf{N} \to \mathbf{R}$.

Definition 1 *Let $g \in \mathcal{F}$. Define,*

$$O(g) = \{ f \in \mathcal{F} \mid \exists B, C > 0,\ \forall n > B,\ f(n) < C\, g(n) \}.$$

And prove the following three facts.

Fact 1 *Let $g \in \mathcal{F}$. Define,*

$$O_1(g) = \{ f \in \mathcal{F} \mid \exists C > 0,\ \forall n,\ f(n) < C\, g(n) \}.$$

Then for any $g \in \mathcal{F}$ such that $g > 0$, and any $f \in \mathcal{F}$,

$$f \in O(g) \iff f \in O_1(g).$$

PROOF: It is clear that $O_1(g) \subseteq O(g)$. Suppose $f \in O(g)$, and fix B and C as required by the definition of $O(g)$. Let,

$$D = \max_{i=0,\ldots,B} \{ f(i)/g(i) \} + 1.$$

This is possible since $g > 0$. Let $C' = \max(C, D)$. For $n > B$,

$$C' g(n) \geq C g(n) > f(n),$$

$\boxed{\text{Burt Rosenberg}}$