Multivariate Polynomial Equations as Matrix Eigenvectors

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Abstract

In this paper, we give an overview of how systems of multivariate polynomial equations may be expressed and solved as matrix eigenvectors. The algorithms are described by the multiplicity table of the stable state associated with the equation system. This table is derived from the Jacobian matrix whose Jacobian matrix is to be found. We consider the case of zero modulus as well as the case of rational zeros. Further numerical results will be treated in separate papers.

1 Introduction

The solution of systems of multivariate polynomial equations was the driving force in the development of algebra from early times to the present. Historically, the invariant theory of algebras [1] or algebras [2] from that time. Thus a different concept of algebraic task arose and the interest in polynomial systems vanished. It has lately been less due to the arrival of computer algebra systems that a renewed interest in this classical subject has arisen, in the last forty years. Recently, a large European project called PINTOP (Polynomial Interpolation) has been initiated to study and implement algorithms for the constructive solution of multivariate polynomial systems.

Meanwhile, polynomial systems have been solved metamorphically through homotopy (intentionally) and local minimization methods, cf. more advanced handbook of Numerical Analysis. However, a good deal of algebraic structures remain unexplored in these approaches. Furthermore, it has to be demonstrated that the number of equations equals that of the unknowns, which is not a natural assumption in polynomial systems.

Also it may be quite difficult - or at least time-consuming - to obtain all solutions, even for not large systems. Therefore, it appears imperative to utilize the algebraic structure of a polynomial system as far as possible.

Through an attempted stability analysis of Buchberger's algorithm, W. van Oeveren and the author of this paper were led to the study of an elimination algorithms which aimed at the solution of a polynomial system, via its bifurcations in terms of a vector eigenvalue problem [3]. Since we were not able to solve the eigenvalue problem for all systems with only isolated zeros, we aborted the efforts (a preliminary report [4] was circulated but not published).

Motivated by the paper [5], We Wunch and his students also attempted to overcome the difficulties described in [3]. By donor considerations, they have expanded in seeking the solution of the elimination algorithms to a point further [6], but have not been successful so far.

A recent meeting in Beijing (July 1987) We Wu and the author came to the
consequence that the desired algorithm would be to incorporate all algebraic techniques of Rukhsberg's algorithm; based on this to start with an extremely large matrix (a matrix of order $m$), and then appeared to us in such a case to cope with Rukhsberg's algorithm for another to get into contact with our approach, i.e. the determination of the system of polynomial equations. This path has been followed in this paper.

In his attempt to generalize the elimination procedure of [3], Weil asked another fundamental limitation. It turns out that for all but a polynomial system into a matrix equation would be required, if the system is not in the triangular form. Therefore, the algebraic problem becomes a complex algebraic problem of the triangular matrix; of course, both these problems have been studied by Krasner's; see also [3]. There, he presented a solution which we will call the characteristic equation of positive dimensions. This equation has been further developed by the author, and a general statement of the problem is given in section 4 of this paper.

In section 3 of this paper, we will explain the new approach to the solution of multivariate polynomial systems in the natural 2 circuits: the circuit from the consideration of the ideal generated by the system of polynomial to the classical one may enable the ideal.

Section 3 will be devoted to a further analysis of this approach in the case where the system has only isolated solutions (i.e., the ideal is 0-dimensional). The consideration of the general problem - gives the same, but the polynomial - gives further insight into the problem.

The elimination problem becomes of zero variables if all zero and some zero from separated algebraic systems of equations will be eliminated in section 4. The algebraic problem must be solved for a separate report to keep this paper to a reasonable length.

Finally, I would like to emphasize that I have regarded the solution of multivariate polynomial system as the system of equations, if the system is 0-dimensional. One see that my approach found a potential for the reduction of the work in the analysis [1]. In a major work, I have made, the problem becomes of course from pure mathematics for the solution of a polynomial problem described in this paper. Therefore, the volume of the memory of I. Galbraith is a good choice for its publication.

2 Representation of a Polynomial System as a Matrix Equation

In [4], polynomial equations are variables, with complex coefficients. We consider the following problem:

Given a set $F$ of $m$ polynomials $f_1, f_2, \ldots, f_m$, determine the set $T = \{t_1, t_2, \ldots, t_m\}$ of all roots $r$ of $f_1, f_2, \ldots, f_m$. Two other words, we want to find all solutions $r \in C$ of the multivariate polynomial system of equations.

\begin{equation}
\begin{aligned}
\forall (x_1, \ldots, x_m) \in R^m, \quad f(x_1, \ldots, x_m) = 0.
\end{aligned}
\end{equation}

It is well known that $T$ is also a set of all zeros of all polynomials in the polynomial ideal $I$ which contains all polynomials of the form $\sum f_i(x_1, \ldots, x_m)$.

\begin{equation}
\begin{aligned}
F = \text{span} \{ (x_1, \ldots, x_m) : f(x_1, \ldots, x_m) = 0 \}.
\end{aligned}
\end{equation}

Then, $F$ may also be characterized by

\begin{equation}
\begin{aligned}
F = \{ x \in C^m : g(x) = 0 \text{ for all } g \in G \}.
\end{aligned}
\end{equation}

Therefore, for the determination of the zeros of $F$, the set $F$ may be replaced by any polynomial set $G \subseteq C^m$ such that

\begin{equation}
\begin{aligned}
F = \text{span} \{ (x_1, \ldots, x_m) : h(x_1, \ldots, x_m) = 0 \}.
\end{aligned}
\end{equation}

is then equivalent to the system of equations (3.1). Each solution of (3.1) is a solution of (3.1) and vice versa.

The transition from (3.1) to an equivalent system (3.1) which admits an easier determination of $F$ has always been the fundamental algebraic approach to the constructive solution of (3.1). Eq. (3.1) can be linearized (3.1) is contained in a linear triangular system (3.5).

Proposition 3.1: If (3.1) is equivalent to (3.1) if

\begin{equation}
\begin{aligned}
\alpha \in \mathbb{C}^m \setminus \{ 0 \}.
\end{aligned}
\end{equation}

where the $i$-th polynomial equation satisfies each $f_i(x) = f_i(x, \ldots, x, x)$ for all zeros $x$ of $f$. (The rank condition in this case is given the set $T$ of the $t_i$ may be different.)

In $F$, let $R = C^m$ be the readily clear case and $F$. If $R$ is a vector space over $C$ (where dimension $n$ is fixed if $F$ is finitely dimensional, otherwise it is infinite).

The standard representation $C^m$ through base of power products (PPG) or monomials $x_1^a_1 \cdots x_m^a_m \subset C^m$ the set of tuples of non-negative integers. Let $1 = e_i = \left[ \begin{array}{c} 1 \ 0 \ \ldots \ 0 \end{array} \right]$. Then $e_i$ forms a basis of $C^m$.
be a semi-FT-base of $R$: then the residue class mod $F$ of each $p \in F^*$ has a unique representation

$$x(p) = \sum_{e \in K} x_e p^e \text{ mod } F, \quad x_e \in C. \quad (2.1)$$

In the representation $R$, the multiplication structure of the ring $R^*$ is described by multiplication tables containing the coefficients $c_{ij} = \phi$ of the representation (2.1). and $F$ of the product $a \cdot p^{e_0} \text{ mod } F = r \cdot p^{e_1}$ (3). (2.2)

$$x_e \cdot x_{e_0} = \sum_{e \in K} c_{e,e_0} x_e \text{ mod } F, \quad r = (e_1)(e_0). \quad (2.4)$$

These $c_{ij}$ permit the recursive computation of the $a_e$ in (2.4) for any $p \in F^*$. For a convolution handling of the multiplication tables, we choose $x(e)$ the polynomial number of these $F$-bases in the set $Z$ in some specified order, and let the matrix $A = (c_{ij}) \in F^{|K|}$, where (2.2) becomes

$$x_e \cdot x_{e_0} = A_{e_0} x_e \text{ mod } F, \quad r = (e_1)(e_0). \quad (2.5)$$

If $F^*$ is a finite-dimensional $A$ and $B[[x]] = \ldots \cdots$ only one finite index of the multiplicative table (cf. section 1.1) of (2.5) will have a non-trivial interpretation in this case.

From the finite $Z$ and the associated multiplication tables matrices $A_e = (c_{ij})$, characterize a semi-FT-base $R$ in $F^*$ completely, and thus there is a one-to-one correspondence between all finite $Z$ in $F^*$ and its residue class $R \cong F^*/F$, the set of equations (cf. (2.10))

$$(A_e - A_{e_0}x)(x_e) = 0, \quad A_e = (c_{ij}). \quad (2.11)$$

must be equivalent to (2.1). Each $x_e \neq 0$ which satisfies (2.11) must also satisfy (2.1) and hence be in the zero set of (2.1), the converse is trivial by (2.10).

From the special structures of (2.11) we obtain our fundamental result:

**Theorem 2.2.** For each zero $z$ of (2.11), the vector $z = \begin{bmatrix} c_{ij} \end{bmatrix} \in C^{|K|}$ is a joint eigenvalue of the matrices $A_e$, with eigenvalue $\phi$, such that, upon suitable normalization, each joint eigenvalue $\phi \in C^{|K|}$ of the $A_e$, with respect to an eigenvalue $\phi \in \phi(z)$, yields a zero $z$ of (2.1) via the interpretation

$$Z = \begin{bmatrix} c_{ij} \end{bmatrix} \phi = (C \phi). \quad (2.12)$$

The zero point eigenvalue is not really restrictive.

**Theorem 2.3.** The multiplication table matrices $A_e$ of (2.11) commute. Indeed, for some $e_0$ the equation $A_{e_0}x_e = A_{e_0}x_{e_0}^e$ and $F$

$$A_{e_0}x_e = A_{e_0}x_{e_0}^e \text{ mod } F, \quad A_{e_0}x_{e_0}^e \quad (2.13)$$

since $Z$ is a basis of this prove the assertion. 0
The $\mathbf{\mu}$ will now be ordered. All relating implementations of the Buchberger algorithm as well as being strictly ordered, with potentially arbitrary nonsense at the end.

In the following, we will acquire a specific admissible order, which is the set of FPs which is kept fixed (onix-sorted otherwise). Such an order defines a total order in the set $\mathbf{B}$ of $n$-tuples of nonnegative integers which are non-negative in the FPs of $\mathbf{B}$. This order is therefore consistent with addition, we will denote it with $\leq$:

\[
\forall \mathbf{a}, \mathbf{b} \in \mathbf{B}, \quad \mathbf{a} \leq \mathbf{b} \iff a_i \leq b_i, \quad \forall i \in \{1, \ldots, n\},
\]

(3.1)

Besides this total order, we will also use the standard lexicographic order $\mathbf{a} \preceq \mathbf{b}$, which is obtained by $\mathbf{a} \leq \mathbf{b}$.

(3.2)

For each polynomial $\mathbf{B}$, we define its leading FP as that FP whose degree is largest in the sense of $\preceq$.

Theorem 3.1 (Buchberger [1]). Let $\mathbf{p} = (p_1, \ldots, p_n)$ be the leading FPs of the polynomials $\mathbf{x}$ in the Grebe Base $\mathbf{G}$ of the polynomial ideal $\mathbf{F} \in \mathbf{B}$. Then $\mathbf{P} = (p_1, \ldots, p_n)$ is not a multiple of some $\mathbf{x}^e$, $e \in \mathbf{B}$.

(3.3)

In a FP basis for a representative $\mathbf{F}$ of the minutiae class $\mathbf{F}$, $\mathbf{B}$ is the monomial

(3.4)

Once a normal form algorithm is used, only the $\mathbf{A}$ for a system $\mathbf{F}$ with rational coefficients will be required.

Corollary 3.2. Let $\mathbf{p} = (p_1, \ldots, p_n)$, be the leading FPs of the polynomials $\mathbf{x} \in \mathbf{G}$ (leading elements), then

\[
J = \mathbf{F} \mathbf{P} = \mathbf{F} \mathbf{P} = \mathbf{B} - \mathbf{G}
\]

(3.5)

In [9], it has also been indicated how the near-degeneracy of a given polynomial system $\mathbf{F}$ may be detected during the execution of the Buchberger algorithm. Further research in this direction may lead to a fast floating-point implementation of the Buchberger algorithm, and of the generation of the multiplication table matrix.

For the purpose of this paper, we will assume that we can determine a PP basis $\mathbf{F}$ of $\mathbf{F}$ and the associated set of $n$-tuples $\mathbf{A}$ such that the polynomial system $\mathbf{F}$ is given in the standard form (3.11). This includes the assumption that we use a total degree ordering of the FPs.

3.2 Relative to Multivariate Polynomial Interpolation

In the case of a b-dimensional ideal $\mathbf{F}$ with the constraint (3.4) has only, indeed solutions, a good deal of additional insight in the constructive solution process may be gained from a consideration of the inverse problem (FP, polynomial interpolation) to see given problem (FP, polynomial system).

(3.6)

Remark: The family $\mathbf{F}$ is $\mathbf{G}$ to be extended to Riemann interpolation and multiple sum cop. [1]. For the sake of simplicity, we restrict ourselves to the case of (3.4) and simple sum at present; some generalizations will be considered in section 3.4.

It is easy to see that, for (3.4), the surface class $\mathbf{F} = \mathbf{B} \mathbf{P} \mathbf{F}$ is represented by a PP-helix $\mathbf{B} \mathbf{P} \mathbf{F}$ and the associated multiplication matrix $\mathbf{A} = (a_{ij})$, plays the same fundamental role as for (3.4).

Take a set $\mathbf{S}$ of $\mathbf{F}$, $\mathbf{F} \mathbf{S} = (a_{ij})$, such that $\mathbf{S}$ is a correct interpolation space for $\mathbf{F}$ which means that

\[
\hat{\mathbf{F}} = \mathbf{F} \mathbf{S}
\]

(3.7)

A set $\mathbf{S}$ which is correct for a given set $\mathbf{F} \mathbf{S}$ be held by the procedure described in [10].
For some fixed $r \in \{1, \ldots, n\}$, we interpolate the functions $u_r \circ \sigma^m$ at $1(1, n)$ on $T$, according to (3.6), to obtain the data

$$v_r = \left( T_r : 1 \right) \left( \sigma^m \right), \quad m = 1(1, n).$$

(3.8)

or, in a shorter notation,

$$\mathbf{v}_r = T_r, \quad V_r.$$ (3.9)

The corresponding solutions of (3.6) are given by $x_{m} \circ \sigma^m(\mathbf{V}_r)$ where the matrix $B_m = (\mathbf{G}_r^m)$ satisfies

$$V_r = \mathbf{N}_r = \mathbf{W}_r = \mathbf{V}.$$ (3.10)

Thus the rows of $\sigma^m(\mathbf{V}_r)$ are $m(1, n)$-th entries of the Vandermonde matrix $V_r$ of (3.7) to the $x$-coordinates of each of the $n$-component matrices $B_m = \mathbf{I}_n$, with eigenvalues $\mathbf{Q}_r^m$ resp.

Transporting (3.10) and using the notation $\mathbf{X}(r)$ as in section 2, we obtain

$$B_r^{-1} \mathbf{X}(r) = (\mathbf{Q}_r^{-m} \mathbf{X}(r), \quad r = 1(1, n).$$

(3.11)

By comparison with Theorem 2.1 and (3.12) we find that the $\mathbf{G}_r^m$ are the multiplication table matrices $\mathbf{B}_m$ of the representations by the basis $T_r$ of the modules class ring $R = \mathbb{Z}[T_r]$. Therefore, by the minimal polynomial $u_r = (\mathbf{A}_m)$, the representation $\mathbf{X}(r)$ as in (3.12),

$$\mathbf{A}_m \mathbf{X}(r) = \mathbf{X}(r).$$

(3.12)

span the required ideal $\mathbf{I} r$ which satisfies (3.3), (6) (3.12).

Thus we have the same set of joint eigenvalues

$$\mathbf{A}_m \mathbf{X}(r) = \mathbf{X}(r), \quad m = 1(1, n).$$

(3.12)

as the central tool for both (FD) and (FD): (3.21)

$$\mathbf{X}(r) \quad \mathbf{X}(r) \quad \mathbf{X}(r) \quad \mathbf{X}(r)$$

(3.11)

(3.12)

$$\mathbf{A}_m \mathbf{X}(r) = \mathbf{X}(r), \quad m = 1(1, n).$$

(3.12)

The duality between (FD) and (FD) has recently been described and analyzed in a more general and abstract form in [8]; we have used it in [8] without knowing of [8].

Now, we want to emphasize the crucial role of the intermediate technique in the routine class ring links to both (FD) and (FD). This establishes the apparent of section 2 as the natural approach to the constructive solution of multivariate polynomial systems by algebraic means.

3.3 Computation of Simple Roots

At first, we consider the case where the $\mathbf{A}_m$ have all joint eigenvalues (as $\mathbf{X}(r) = \mathbf{X}(r)$) when $n = \deg(\sigma)$ is a member of some set of $\mathbf{X}(r)$. Specifying an eigenvalue $\mathbf{A}_m$ represents our $\mathbf{X}(r)$ and new ones. Obviouly, all $\mathbf{A}_m$ are diagonalizable.

If the IF groups $\mathbf{A}_m$ are contained in all $\mathbf{X}(r)$, we use $\mathbf{Q}_r$ or $\mathbf{Q}_0$ (if $\mathbf{Q}_r$ is not trivial) in $\mathbf{Q}_0$. Therefore, we may obtain the res. component of the sum of first components of $\mathbf{Q}_r^m$ components of $\mathbf{A}_m$.

$$\mathbf{Q}_r^m = \mathbf{Q}_r^m \mathbf{Q}_r^m = \mathbf{A}_m \mathbf{A}_m$$

(3.14)

If all $\mathbf{Q}_r^m$ are diagonalize, a diagonalizes $\mathbf{A}_m$ can only occur once. We may have shown $\mathbf{A}_m$ such that the compositions $\mathbf{A}_m$ are not all diagonal, say $\mathbf{A}_m \in \mathbf{Q}_r^m$. Then $\mathbf{Q}_r^m$ will be a multiple eigenvalue $\mathbf{A}_m$ will all eigenvalues(3.12) in equal to the number of times $\mathbf{Q}_r^m$-components coincide, and there will be an all eigenvalues of dimension $r$. Thus the individual in diagonal can not be identified.

In the case we may check a linear combination of basis vectors of this eigenvalue (with 'first component')

$$\sum_{r = 1(1, n)} \mathbf{a}_r = (a_1, \ldots, a_n) \quad \mathbf{Q}_r \quad \mathbf{Q}_r$$

(3.33)

against some other $\mathbf{A}_m$.

$$\mathbf{A}_m \mathbf{a}_r = \mathbf{a}_r, \quad \mathbf{a}_r = \mathbf{a}_r.$$ (3.33)

$$\mathbf{a}_r$$

(3.33)

Thus, the coefficients $\mathbf{a}_r$ in (3.33) may be obtained from an $n$-component eigenvalues (3.17).

Finally, one may wish to avoid this complication by selecting an $\mathbf{A}_m$ with distinct eigenvalues of $L$. There may be no coordinate direction which separates all zeros. The solutions may be described by means of particular 2.1. From (2.11) is equivalent to (3.11) and to the semi-Gröbner basis system (2.1) there are $n$-component $\mathbf{C}_r$ such that

$$\mathbf{A}_m \mathbf{C}_r = \mathbf{C}_r \mathbf{A}_m \sigma^m(\mathbf{a}_r) = \mathbf{a}_r$$

(3.18)

If $\mathbf{C}_r \mathbf{a}_r = \mathbf{a}_r$ then $\mathbf{A}_m$ must generate a zero of $\mathbf{a}_r$ i.e. there can not be redundant eigenvalues. Again, there may be such $\mathbf{a}_r$ although the complete $\mathbf{a}_r$ is not simple $\mathbf{a}_r$ always have such $\mathbf{a}_r$. If all $\mathbf{C}_r$ are composed together with the multiplication table matrices $\mathbf{A}_m$ it should not be difficult to select an appropriate $\mathbf{a}_r$. Remember that every zero of the $\mathbf{C}_r$ is of (4.1) last paragraph of section 3).

The actual solution of the eigenvalues (3.13) should generally have to be done by a numerical approximation algorithm. Suitable software may be found in the NAG and IMSL mathematical software libraries or in LAPACK (11).
Example 3.2. Consider the system
\[ \begin{align*}
\dot{x} & = y \\
y & = -x + u
\end{align*} \]
with the semiconfined Gaussian basis system
\[ \begin{align*}
x & = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \\
y & = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \\
u & = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}
\end{align*} \]
According to Theorem 3.1, \( F = \begin{bmatrix} I & 0 \end{bmatrix} \), \( \gamma = 0 \), \( n = 0 \), and \( m = 0 \).

The semiconfined principal value is \( \{0, 0, 0, 0\} \) as well as \( \gamma \).

Find \( \delta = \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} \) which satisfies (3.20).
as seen below and preserve it for all larger sections. Consider such a finite rank-k series of \( (4.1) \).

Let \( a_0(x) = z_1 \) be the \( z_1 \) th smallest eigenvalue of \( z_1 \) in \( Z \) for some integer \( s \), and let \( \varphi(x) \) be its unique solution of \( z_1(x) \), s.t. \( z_1(x) \), s.t. \( z_1(x) \).

Thus we have proved

**Theorem 6.1.** In the case of a complex-valued linear operator \( F \) that is continuous and bounded on \( H \), the operator \( F \) is compact if and only if \( F \) is finite-dimensional. Moreover, if \( F \) is finite-dimensional, then \( \ker(F \cdot -\lambda I) = \{0\} \) for all \( \lambda \) in \( \mathbb{C} \).

As a consequence, we have that \( \ker(F \cdot -\lambda I) = \{0\} \) for all \( \lambda \) in \( \mathbb{C} \).

Note that if we have dropped the assumption that \( F \) is finite-dimensional, then \( \ker(F \cdot -\lambda I) = \{0\} \) for all \( \lambda \) in \( \mathbb{C} \).

**Corollary 6.2.** The infinite-dimensional operator \( F \) is compact if and only if \( \ker(F \cdot -\lambda I) = \{0\} \) for all \( \lambda \) in \( \mathbb{C} \).

For the practical application of this result, there remains to be discussed the following:

(i) We must determine suitable finite sections \( S \mathcal{P} \) and \( S \mathcal{P} \) that are close to the \( S \mathcal{P} \) base vector of \( S \mathcal{P} \), with \( \mathcal{P} \) and \( \mathcal{P} \) as small as possible.

(ii) We must compute all \( S \mathcal{P} \) solutions \( L \) in \( \mathcal{P} \) of the \( S \mathcal{P} \) eigenvalue problem.

(iii) We must compute the coefficients \( \lambda \) in \( \mathcal{P} \) of the \( S \mathcal{P} \) eigenvalue problem.

(iv) We must compute the coefficients \( \lambda \) in \( \mathcal{P} \) of the \( S \mathcal{P} \) eigenvalue problem.

(v) We must compute the coefficients \( \lambda \) in \( \mathcal{P} \) of the \( S \mathcal{P} \) eigenvalue problem.

For each \( \lambda \), the nontrivial solutions \( \lambda \) for \( \lambda \) must be made more efficient; we will not discuss the algorithms here (cf. eq. (5)). The principal elements of the algorithmic solution of \( (4.1) \) will be described in the following section.

Example 6.1. Consider the ideal \( F \) defined by the graph matrix

\[
p(x, y) = \begin{cases} 1 & \text{if } (x, y) \in E \\ 0 & \text{otherwise} \end{cases}
\]

According to Theorem 6.1, the infinite-dimensional operator \( F \) is compact if and only if \( \ker(F \cdot -\lambda I) = \{0\} \) for all \( \lambda \) in \( \mathbb{C} \).

**5.2 Algorithmic Solution of Singular Eigenproblems**

We consider a so-called singular eigenproblem

\[
(\mathcal{A} - \lambda \mathcal{I}) \mathbf{x} = 0,
\]

with \( \mathcal{A} \in \mathbb{C}^{n \times n}, \lambda \in \mathbb{C}, \mathbf{x} \in \mathbb{C}^n \), and \( \mathcal{I} \in \mathbb{C}^{n \times n} \).

The solution \( \lambda \) of \( \det(\mathcal{A} - \lambda \mathcal{I}) = 0 \) for \( \lambda \) in \( \mathbb{C} \).

Note that for \( \lambda \) in \( \mathbb{C} \), the determinant of \( \det(\mathcal{A} - \lambda \mathcal{I}) = 0 \) for \( \lambda \) in \( \mathbb{C} \).

For the practical application of this result, there remains two tasks which we must be able to perform algorithmically:

(i) We must determine suitable finite sections \( S \mathcal{P} \) and \( S \mathcal{P} \) that are close to the \( S \mathcal{P} \) base vector of \( S \mathcal{P} \), with \( \mathcal{P} \) and \( \mathcal{P} \) as small as possible.

(ii) We must compute all \( S \mathcal{P} \) solutions \( L \) in \( \mathcal{P} \) of the \( S \mathcal{P} \) eigenvalue problem.

(iii) We must compute the coefficients \( \lambda \) in \( \mathcal{P} \) of the \( S \mathcal{P} \) eigenvalue problem.

(iv) We must compute the coefficients \( \lambda \) in \( \mathcal{P} \) of the \( S \mathcal{P} \) eigenvalue problem.

(v) We must compute the coefficients \( \lambda \) in \( \mathcal{P} \) of the \( S \mathcal{P} \) eigenvalue problem.

For each \( \lambda \), the nontrivial solutions \( \lambda \) for \( \lambda \) must be made more efficient; we will not discuss the algorithms here (cf. eq. (5)). The principal elements of the algorithmic solution of \( (4.1) \) will be described in the following section.
Thus the complete solution set of (4.8) consists of the general epigenesis

\[ k(x) = \sum_{i=1}^{n} \alpha_i \lambda_i(x), \quad \text{with } \lambda_i \in \mathbb{R}^n, \]

and \( n = \dim \{ \lambda_i \}. \)

Example 4.1: The compact algorithm for the computation of the \( k(x) = \lambda(x) \cdot n \) algorithm is the general epigenesis that has been derived by the author, which is incorporated by the method 2, transdifferentiated. The algorithm and its implementation will be described in a separate paper [15].

It should be noted that the \( k(x) = \lambda(x) \cdot n \) algorithm is a generalization of a family of linear processes with complex-valued coefficients. The complex-valued process \( k(x) \) is determined as a function of \( \lambda_i(x) \) and \( \lambda_i(x) \) is the eigenfunction of \( \lambda(x) \cdot n \).

Theorem 4.2: (a) The complete solution of (4.8) is

\[ \lambda(x) = \sum_{i=1}^{n} \alpha_i \lambda_i(x), \]

with \( \alpha_i \in \mathbb{R} \).

(b) The general solution of (4.8) is

\[ \lambda(x) = \sum_{i=1}^{n} \alpha_i \lambda_i(x), \]

with \( \alpha_i \in \mathbb{R} \).

Theorem 4.3: (a) The complete solution of (4.8) is

\[ \lambda(x) = \sum_{i=1}^{n} \alpha_i \lambda_i(x), \]

with \( \alpha_i \in \mathbb{R} \).

(b) The general solution of (4.8) is

\[ \lambda(x) = \sum_{i=1}^{n} \alpha_i \lambda_i(x), \]

with \( \alpha_i \in \mathbb{R} \).

Theorem 4.4: (a) The complete solution of (4.8) is

\[ \lambda(x) = \sum_{i=1}^{n} \alpha_i \lambda_i(x), \]

with \( \alpha_i \in \mathbb{R} \).

(b) The general solution of (4.8) is

\[ \lambda(x) = \sum_{i=1}^{n} \alpha_i \lambda_i(x), \]

with \( \alpha_i \in \mathbb{R} \).

Theorem 4.5: (a) The complete solution of (4.8) is

\[ \lambda(x) = \sum_{i=1}^{n} \alpha_i \lambda_i(x), \]

with \( \alpha_i \in \mathbb{R} \).

(b) The general solution of (4.8) is

\[ \lambda(x) = \sum_{i=1}^{n} \alpha_i \lambda_i(x), \]

with \( \alpha_i \in \mathbb{R} \).

Theorem 4.6: (a) The complete solution of (4.8) is

\[ \lambda(x) = \sum_{i=1}^{n} \alpha_i \lambda_i(x), \]

with \( \alpha_i \in \mathbb{R} \).

(b) The general solution of (4.8) is

\[ \lambda(x) = \sum_{i=1}^{n} \alpha_i \lambda_i(x), \]

with \( \alpha_i \in \mathbb{R} \).

Theorem 4.7: (a) The complete solution of (4.8) is

\[ \lambda(x) = \sum_{i=1}^{n} \alpha_i \lambda_i(x), \]

with \( \alpha_i \in \mathbb{R} \).

(b) The general solution of (4.8) is

\[ \lambda(x) = \sum_{i=1}^{n} \alpha_i \lambda_i(x), \]

with \( \alpha_i \in \mathbb{R} \).

Theorem 4.8: (a) The complete solution of (4.8) is

\[ \lambda(x) = \sum_{i=1}^{n} \alpha_i \lambda_i(x), \]

with \( \alpha_i \in \mathbb{R} \).

(b) The general solution of (4.8) is

\[ \lambda(x) = \sum_{i=1}^{n} \alpha_i \lambda_i(x), \]

with \( \alpha_i \in \mathbb{R} \).

Theorem 4.9: (a) The complete solution of (4.8) is

\[ \lambda(x) = \sum_{i=1}^{n} \alpha_i \lambda_i(x), \]

with \( \alpha_i \in \mathbb{R} \).

(b) The general solution of (4.8) is

\[ \lambda(x) = \sum_{i=1}^{n} \alpha_i \lambda_i(x), \]

with \( \alpha_i \in \mathbb{R} \).

Theorem 4.10: (a) The complete solution of (4.8) is

\[ \lambda(x) = \sum_{i=1}^{n} \alpha_i \lambda_i(x), \]

with \( \alpha_i \in \mathbb{R} \).

(b) The general solution of (4.8) is

\[ \lambda(x) = \sum_{i=1}^{n} \alpha_i \lambda_i(x), \]

with \( \alpha_i \in \mathbb{R} \).
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